Perfect matchings a.e. and generically

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Based on joint work with Kun and Sabok and Poulin and Zomback

Classical results

Theorem (Kőnig)

Every d-regular bipartite graph has a perfect matching.

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- $\sigma = \frac{1}{d}$ is a fractional perfect matching.
- Let $F(\sigma) = \sigma^{-1}(0,1)$.
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Can we find definable analogs of the above theorem?

Counterexamples

- Irrational rotation graph has no Borel pm a.e./generically.
- (Conley, Jackson, Marks, Seward, Tucker-Drob) There are acyclic d-regular Borel graphs with no Borel p.m. for any $d \geq 2$.
- (Kun) There are d-regular acyclic pmp Borel graphs with no Borel pm a.e. for all $d \geq 2$.

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Theorem (B., Kun, Sabok)

Every d-regular hyperfinite one-ended bipartite pmp Borel graph has a Borel pm a.e.

Theorem (B., Poulin, Zomback)

Every d-regular one-ended bipartite Borel graph has a Borel pm generically.

toasts

Definition

A borel family of sets $\mathcal{T} \subset V(G)^{<\infty}$ is a **toast** if it satisfies properties (1) and (2) of the below definition, and it is a **connected toast** if it also satisfies property 3:

1. $\bigcup_{K \in \mathcal{T}} E(K) = E(G)$,
2. for every pair $K, L \in \mathcal{T}$ either $(N(K) \cup K) \cap L = \emptyset$ or $K \cup N(K) \subseteq L$, or $L \cup N(L) \subseteq K$,
3. for every $K \in \mathcal{T}$ the induced subgraph on $K \setminus \bigcup_{K \supsetneq L \in \mathcal{T}} L$ is connected.

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Theorem (B., Kun, Sabok)

Every one-ended hyperfinite Borel graph admits a connected toast a.e.

Theorem (B., Poulin, Zomback)

Every one-ended Borel graph admits a connected toast generically.

PM generically

Theorem (BPZ)

Any one-ended bipartite d-regular Borel graph admits a Borel perfect matching generically.

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Any one-ended bipartite d-regular Borel graph admits a Borel perfect matching generically.

Let $\mathcal{T}$ be a connected toast. For every $L \in \mathcal{T}_1$ there is an $m \in \omega$, an $L \subset K \in \mathcal{T}_{<m}$, and a fractional matching σ' such that

1. $\sigma'(e) \in \{0, 1\}$ for all $e \in E(L)$.
2. $\sigma'(e) = \sigma(e)$ for all $e \notin E(K)$.

PM a.e.

Lemma

Given a measurable fpm σ, we can find a measurable σ' with $\sigma'(e) \in \{0, \frac{1}{2}, 1\}$ and no cycles.

In fact, any extreme point in the space of measurable fpm has this property.

PM a.e.

Lemma

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In fact, any extreme point in the space of measurable fpm has this property.

Let R be the set of such matchings and given $\sigma \in R$ let $L(\sigma) = \sigma^{-1}(\frac{1}{2})$.

Improving matchings

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Proof.

We can assume $\sigma' \in R$ by the Choquet–Bishop–de Leeuw theorem and convexity.
Let $A = \{e \in E(G) \setminus L(\sigma) : \sigma'(e) \neq \sigma(e)\}$ and
$B = \{e \in L(\sigma) : \sigma'(e) \neq \sigma(e)\}$.
We know $L(\sigma') \setminus L(\sigma) \subseteq A$ and $B = L(\sigma) \setminus L(\sigma')$.
Also,
$\mu(A) \leq 2 \int_{E(G) \setminus L(\sigma)} |\sigma' - \sigma|$ and $\mu(B) = 2 \int_{L(\chi)} |\sigma' - \frac{1}{2}| = 2 \int_{L(\sigma)} |\sigma' - \sigma|$.
Putting this together gives what we want.

□

Improving a matching

Let k be large and $\lambda > \varepsilon > 0$ depending on k be tiny. Using a toast find k Borel families of cycles $\mathcal{C}_1, \ldots, \mathcal{C}_k$ each consisting of pairwise edge-disjoint cycles such that

1. every edge not in L is covered by at most one cycle of $\bigcup_{i=1}^k \mathcal{C}_i$,
2. $\mu(\bigcap_{i=1}^k E(\bigcup \mathcal{C}_i) \cap E(L)) > \frac{1}{2}\mu(L)$,

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Given this, let $\chi = \frac{\lambda}{d} + (1 - \lambda\sigma)$.

Flip a coin to decide if we'll add or subtract ε alternating around each cycle in χ and let the result be σ'.

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Then $|\sigma'(e) - \sigma(e)| < \lambda + \varepsilon$ for $e \notin L(\sigma)$

and

$\mathbb{E}|\sigma' - \sigma| = \Omega(\varepsilon\sqrt{k})$ for $e \in \bigcap_{i=1}^k E(\bigcup \mathcal{C}_i)$ by Stirling's approximation.

Problems

- Does every one-ended bipartite Borel graph satisfy $\chi'_{BM} \leq \Delta(G)$?
- Does every bipartite d-regular Borel graph that admits a connected toast have a Borel perfect matching?